# IIR Filter Design

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- IIR Filter Design
- Impulse Invariance
- Bilinear Transformation

# Filter Design Steps

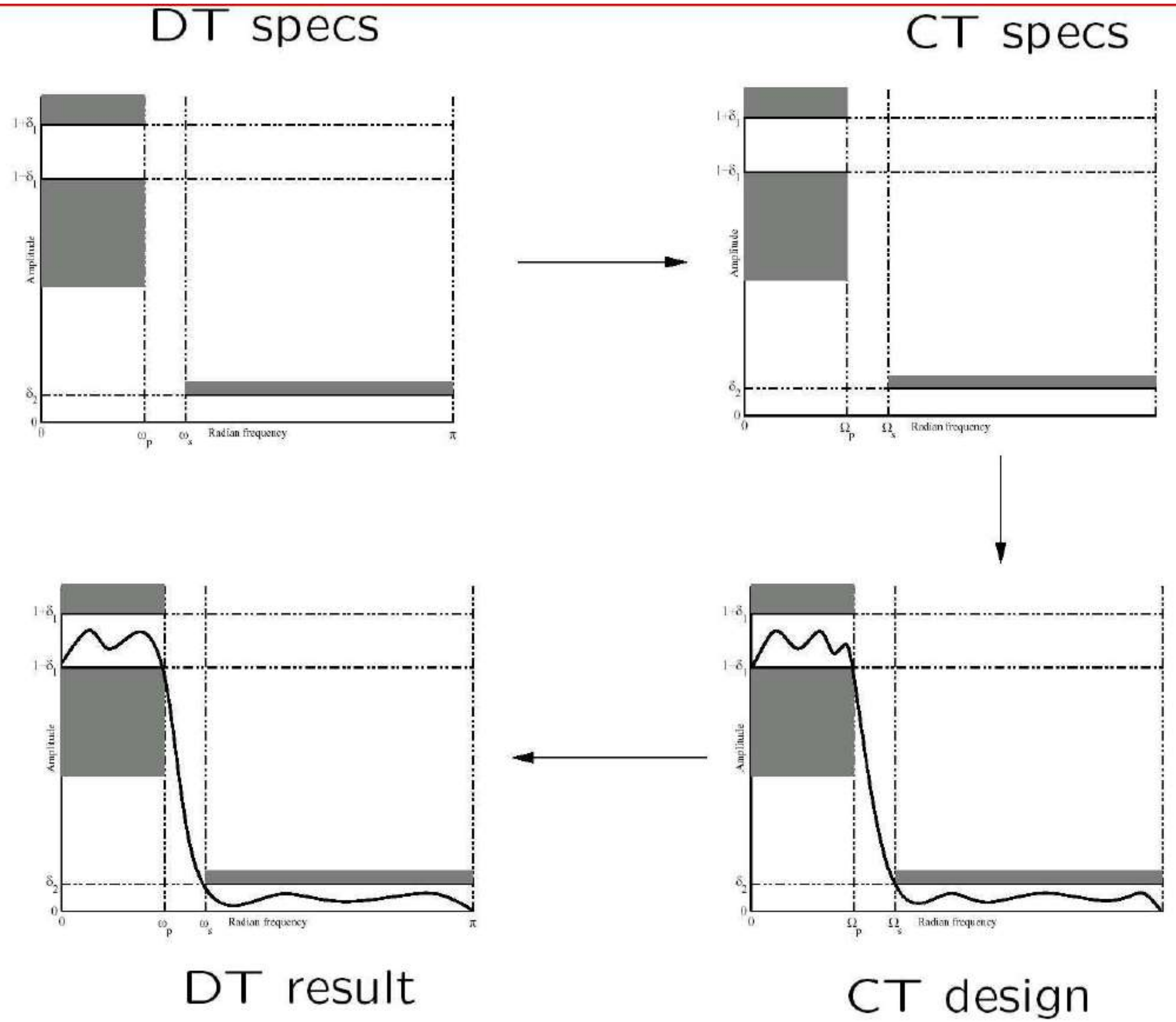
- The design of Digital Filter involves five steps:
  - Specifications according to filter requirements.
  - Calculations of suitable filter coefficients
  - Representation of filter by a suitable structure (realization)
  - Analysis of the effects of finite word length on filter performance.
  - Implementation of filter in software and/or hardware.

## Discrete Time filter Design (Coefficient calculation)

- Discrete-time IIR filter design is done using analog filter techniques:

1. Analog IIR filter design methods have simple closed form solutions;
2. Design examples have existed for years.
3. Direct design of IIR filters has traditionally been avoided
4. Direct design of FIR filters is possible.

# DT IIR filter design (Coefficient calculation)

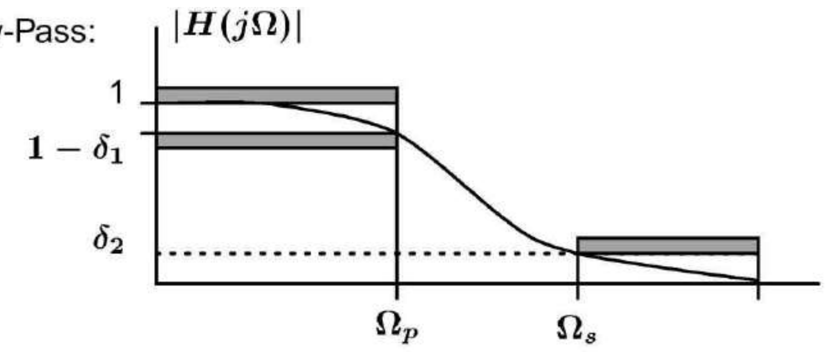


# Traditional Analog Filter Design

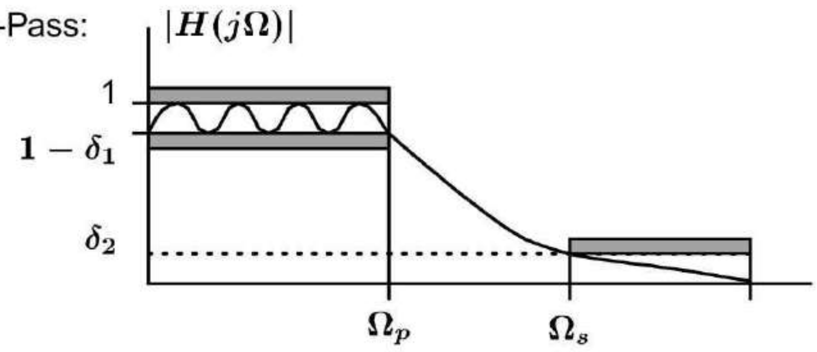
- Butterworth filters
- Chebyshev filters
- Elliptic filters

# Traditional Analog Filter Design

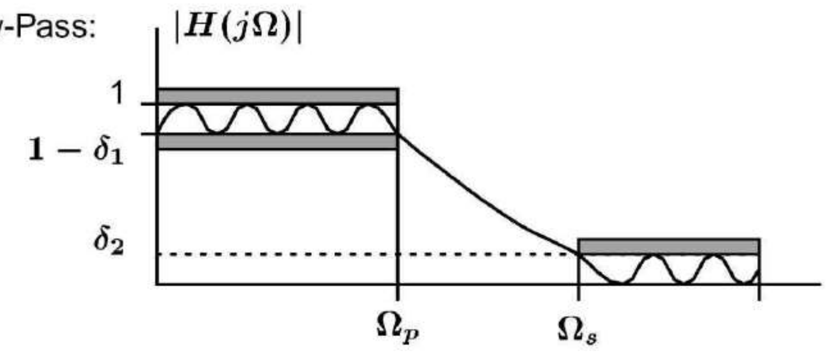
Butterworth Low-Pass:




Chebyshev (I) Low-Pass:




Elliptic Low-Pass:



# Butterworth Design

C A S

$$|H_c(j\Omega)|^2 = \frac{1}{1 + \left(\frac{j\Omega}{j\Omega_c}\right)^{2N}} = H_c(j\Omega)H_c^*(j\Omega) = H_c(j\Omega)H_c(-j\Omega)$$

$$H(s)H(-s) = \frac{1}{1 + \left(\frac{s}{j\Omega_c}\right)^{2N}}$$

$$s_k = (-1)^{1/2N} j\Omega_c = \Omega_c e^{j\left(\frac{\pi}{2} + \frac{\pi}{2N} + \frac{k\pi}{N}\right)}, \quad k = 0, 1, \ldots, 2N-1$$

To get a stable and causal filter,

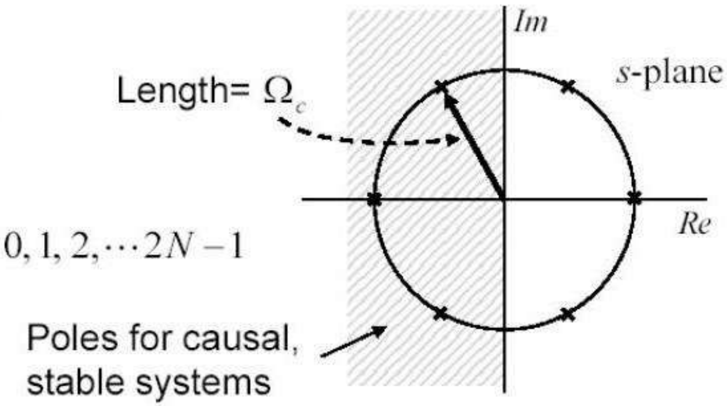
choose $H_c(s)$ to implement the poles in the left-hand plane.

# Butterworth Design

$$|H_c(j\Omega)|^2 = \frac{1}{1+(j\Omega/j\Omega_c)^{2N}} \quad \Leftarrow \quad H_c(s)H_c(-s) = \frac{1}{1+(s/j\Omega_c)^{2N}}$$

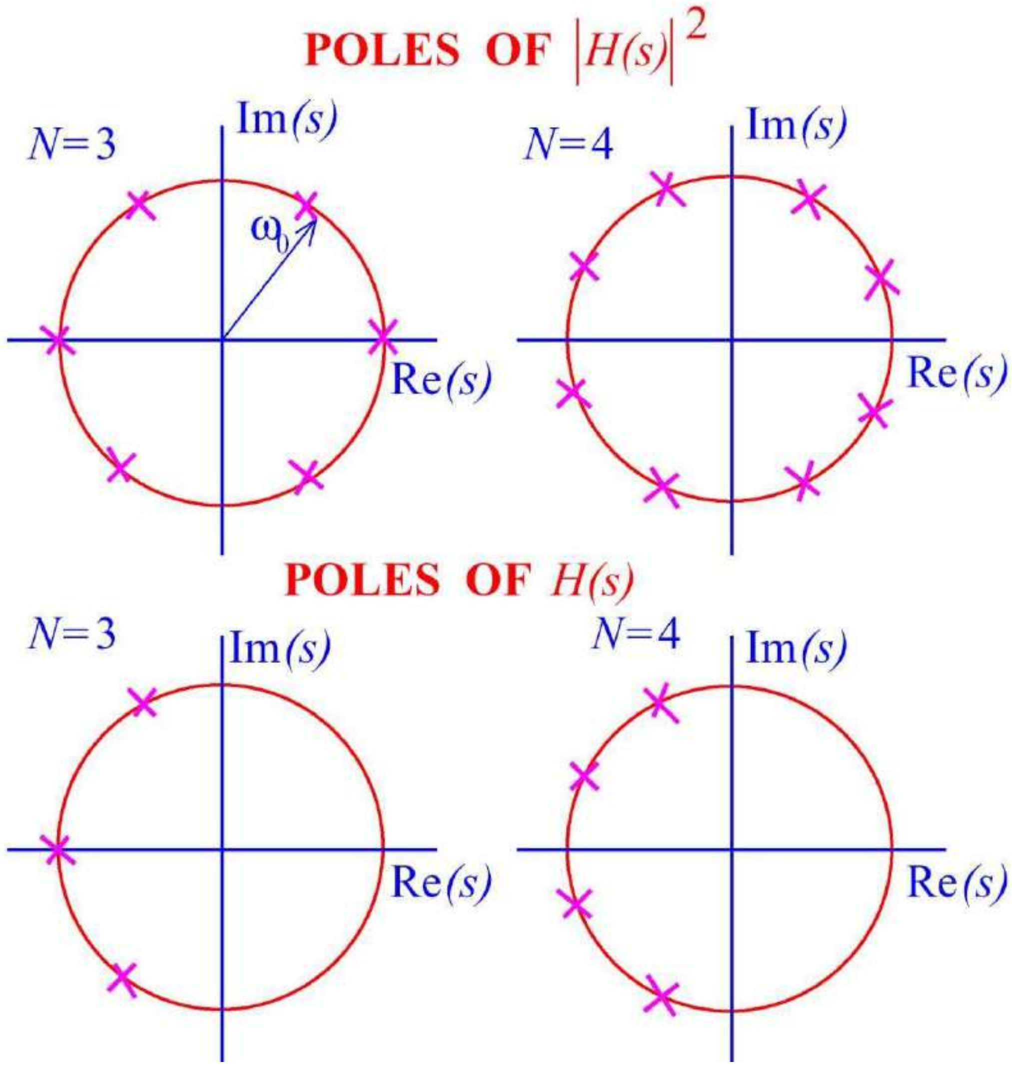
Poles are at the roots of $\quad 1+(s_k/j\Omega_c)^{2N} = 0$

$$\begin{aligned} s_k &= (-1)^{1/2N}(j\Omega_c) \\ &= \Omega_c \exp\left\{\frac{j\pi}{2N}(2k-1)\right\}\exp\left\{\frac{j\pi}{2}\right\} \\ &= \Omega_c \exp\left\{\frac{j\pi}{2N}(2k+N-1)\right\} \quad k = 0, 1, 2, \cdots 2N-1 \end{aligned}$$

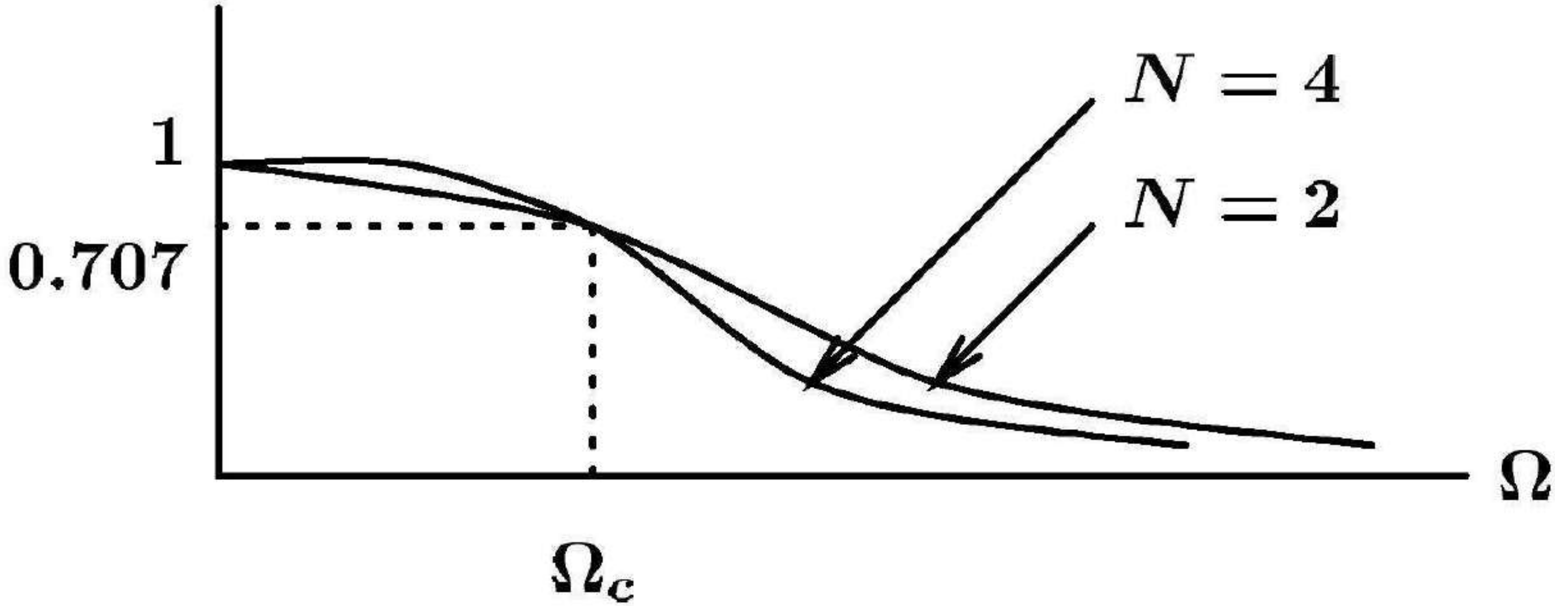



$$H_c(s) = \frac{1}{\prod (s-s_k)(s-s_k^*)}$$ where $s_k^*$ and $s_k^*$ are on the left plane

# Butterworth Design



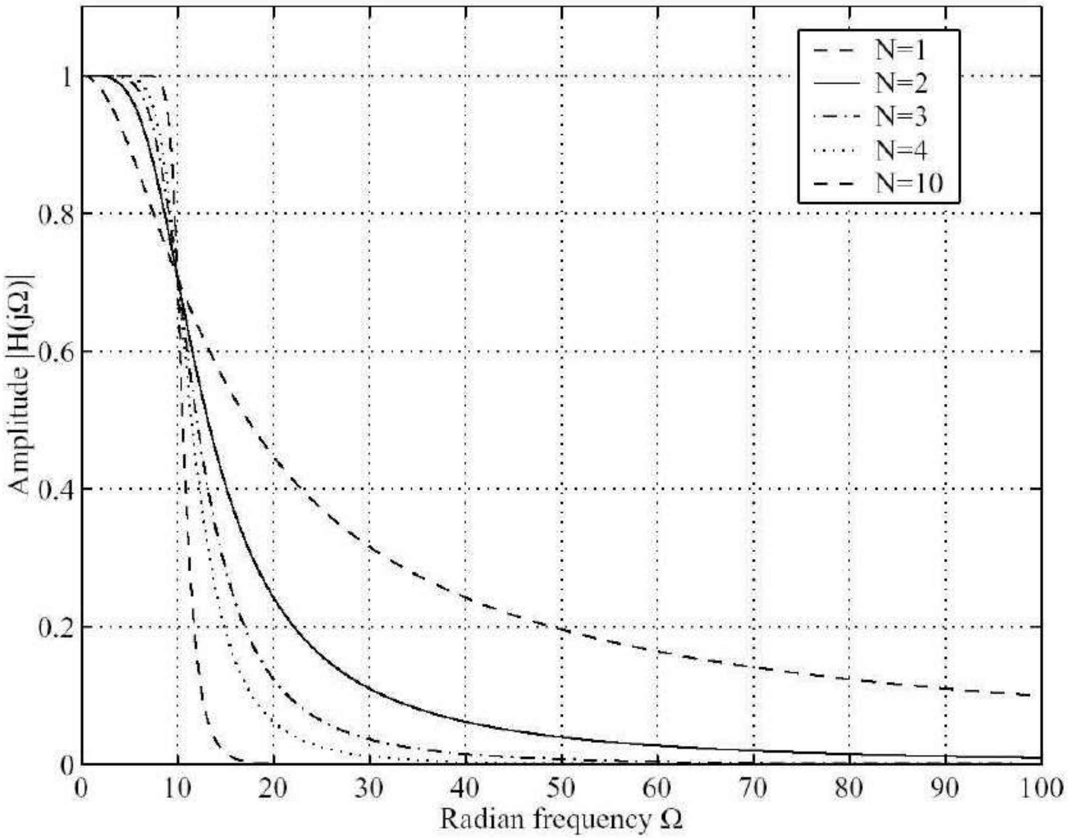
# Butterworth Design




Butterworth filters are fully specified by the cutoff frequency $\Omega_c$ and the filter order $N$

# Butterworth Design

$$|H(j\Omega)|^2 = \frac{1}{1 + (j\Omega/j\Omega_c)^{2N}}$$

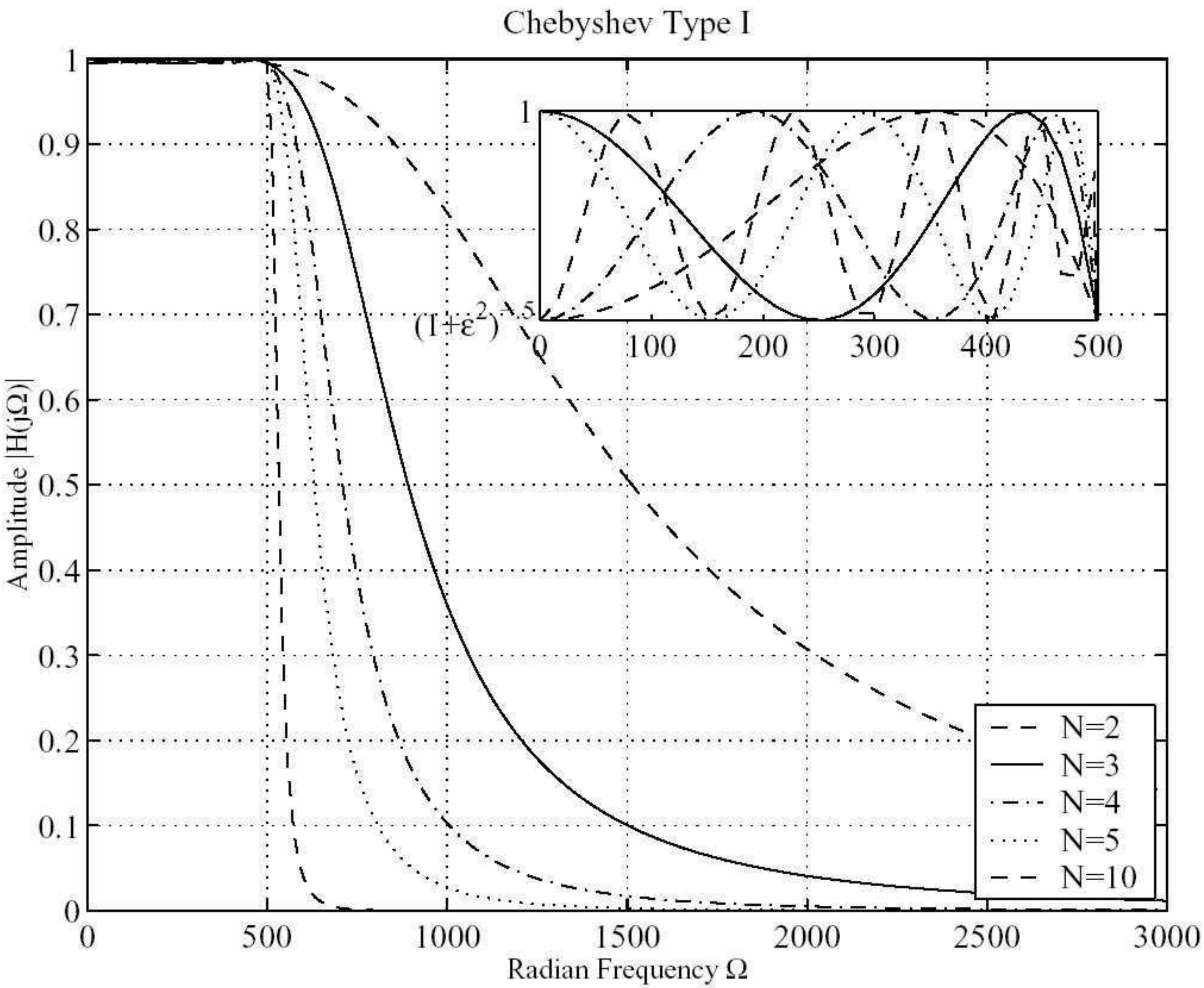



- Monotonic
- poles: $s = (-1)^{1/2N} j\Omega_c$

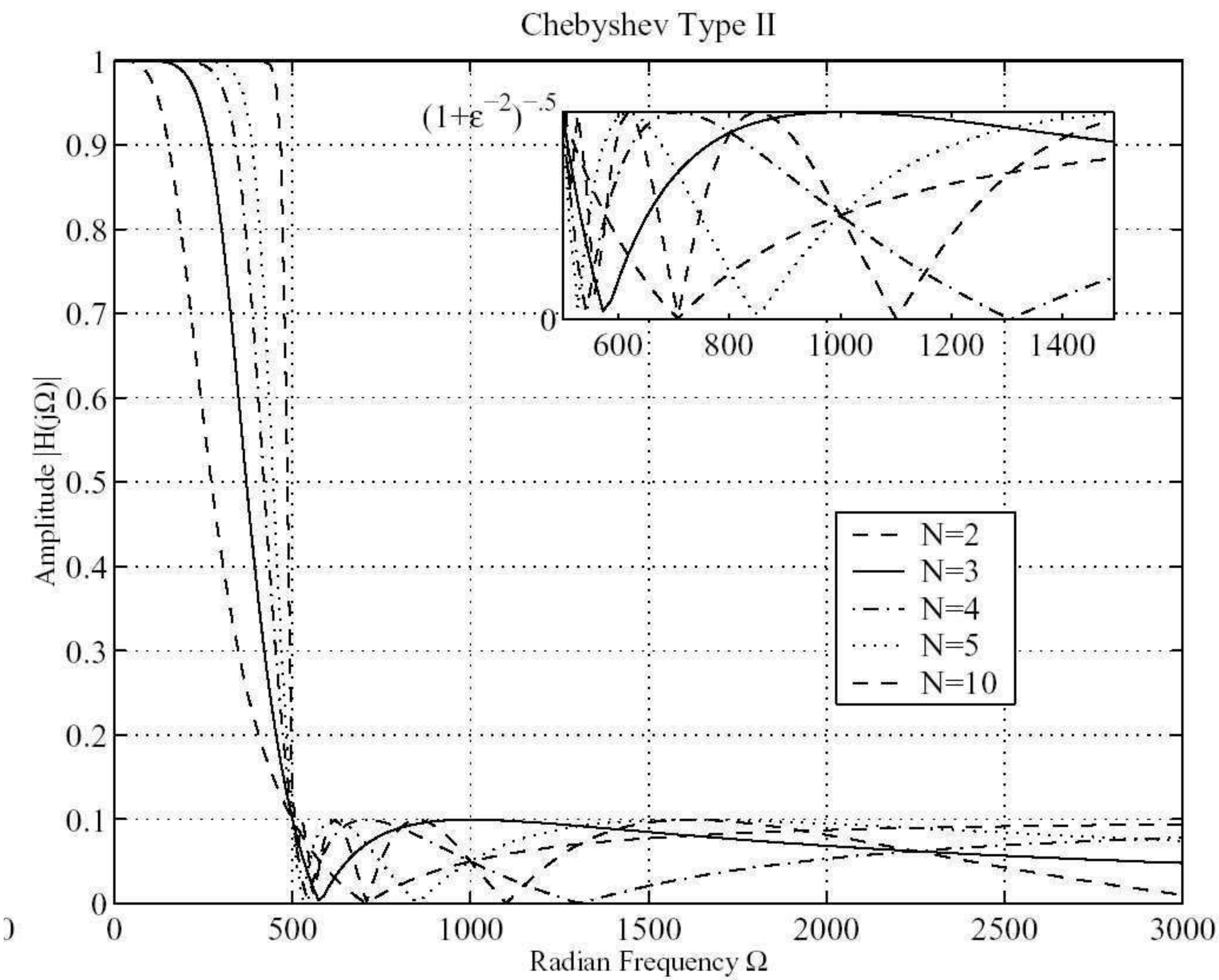
# Chebyshev filters

| | Type I | Type II |
|---|---|---|
| Def | $\|H(j\Omega)\|^2 = \frac{1}{1+\varepsilon^2 T_N^2(\Omega/\Omega_c)}$ | $\|H(j\Omega)\|^2 = \frac{1}{1+(\varepsilon^2 T_N^2(\Omega_c/\Omega))^{-1}}$ |
| Passband | Ripple from 1 to $\sqrt{1/1+\varepsilon^2}$ | Monotonic |
| Stopband | Monotonic | Ripple $\sqrt{1/(1+1/\varepsilon^2)}$ |

# Chebyshev Type I



# Chebyshev Type II



# IIR filter design (Coefficient calculation)

- Most widely used methods:
  - Pole-zero placement
  - Impulse Invariance
  - Bilinear Transform

# 1. IIR Filter design: Impulse-Invariant Method

- **basic principle: sampling of impulse response of an analogue filter**,
- mapping: $H_A(p)$ -> $H(z)$,
- resulting filter implementation as a parallel bank of two-pole filter,
- aliasing effect following from sampling process and its impact,

# Impulse-Invariant Method (Impulse Invariant Transformation)

**Objective:** to design an IIR filter having an impulse response *h(n)* as the sampled version of the impulse response of the analogue filter $h_A(t)$:

$$h_A(t) = h_A(nT) = h(nT) = h(n) \quad n = 0,1,2,\ldots$$

where $T$ is the sampling interval.

In consequence of this result, the frequency response of the digital filter is an aliased version of the frequency response of the corresponding analogue filter.

Let the transfer function of the analogue filter be given:

$$H_A(p) = \frac{L[y(t)]}{L[x(t)]} = \frac{B(p)}{A(p)} = \frac{\sum_{k=0}^{M} b_k p^k}{\sum_{k=0}^{N} a_k p^k}$$

Let us assumed that the order $M$ of the numerator is less that the order $N$ of the denominator and that all poles of $H_A(p)$ are simple. If the poles of $H_A(p)$ are not simple, the discussion in this section can be appropriately modified. Then, we rewrite the transfer function of the analogue filter in its partial expansion, as follows

$$H_A(p) = \frac{\sum_{k=0}^{M} b_k p^k}{\sum_{k=0}^{N} a_k p^k} = \sum_{k=1}^{N} \frac{c_k}{p + d_k}$$

where $(-d_k)$ is the location of the $k$-th pole and

$$c_k = H_A(p)(p + d_k)\Big|_{p=-d_k}$$

## The impulse response of the analogue filter $h_A(t)$ :

$$h_A(t) = L^{-1}\left[H_A(p)\right] =$$

$$= L^{-1}\left[\sum_{k=1}^{N} \frac{c_k}{p + d_k}\right] = \sum_{k=1}^{N} L^{-1}\left[\frac{c_k}{p + d_k}\right] =$$

$$= \sum_{k=1}^{N} c_k e^{-d_k t} \rightarrow t = nT$$

## The impulse response of the digital filter *h(nT)*:

$$h(n) = h(nT) = h_A(nT) = \sum_{k=1}^{N} c_k e^{-d_k nT}$$

$$n = 0,1,2,3,\ldots,\infty$$

# Transfer function of the digital filter:

$$H(z) = Z\left[h(n)\right] =$$

$$= \sum_{n=0}^{\infty} h(n) z^{-n} =$$

$$= \sum_{n=0}^{\infty} z^{-n} \sum_{k=1}^{N} c_k e^{-d_k nT} =$$

$$= \sum_{k=1}^{N} c_k \sum_{n=0}^{\infty} \left(e^{-d_k T} z^{-1}\right)^n =$$

$$= \sum_{k=1}^{N} c_k \frac{1}{1 - e^{-p_k T} z^{-1}}$$

**Transfer function of the analogue filter:**

$$H_A(p) = \sum_{k=1}^{N} \frac{c_k}{p + d_k}$$

**Transfer function of the digital filter:**

$$H(z) = \sum_{k=1}^{N} \frac{c_k}{1 - e^{-d_k T} z^{-1}}$$

Comparing $H_A(p)$ and $H(z)$ it can be seen that $H(z)$ can be obtained from $H_A(p)$ by using the mapping relation:

$$\frac{c_k}{p + d_k} \rightarrow \frac{c_k}{1 - e^{-d_k T} z^{-1}}$$

With the previous given expressions for the transfer function *H(z)*, **the IIR filter is easily realized as a parallel bank of single-pole filters:**

$$H(z) = \sum_{k=1}^{N} \frac{c_k}{1 - e^{-d_k T} z^{-1}}$$

With the previous given expressions for the transfer function *H(z)*, **the IIR filter is easily realized as a parallel bank of single-pole filters.**

If some of poles are complex-valued, they may be paired together and **combined to form two-pole filter sections with real-valued coefficients:**

$$\frac{c_k}{p+d_k} \rightarrow \frac{c_k}{1-z^{-1}e^{-d_k T}}$$

$$\frac{\overline{c_k}}{p+\overline{d_k}} \rightarrow \frac{\overline{c_k}}{1-z^{-1}e^{-\overline{d_k} T}}$$

$$\frac{c_k}{p+d_k} + \frac{\overline{c_k}}{p+\overline{d_k}} \rightarrow \frac{c_k}{1-z^{-1}e^{-d_k T}} + \frac{\overline{c_k}}{1-z^{-1}e^{-\overline{d_k} T}}$$

With the previous given expressions for the transfer function *H(z)*, **the IIR filter is easily realized as a parallel bank of single-pole filters.**

If some of poles are complex-valued, they may be paired together and **combined to form two-pole filter sections with real-valued coefficients.**

In addition, two factors containing real-valued poles may be **combined to form two-pole filters with real-valued coefficients.**

With the previous given expressions for the transfer function *H(z)*, **the IIR filter is easily realized as a parallel bank of single-pole filters.**

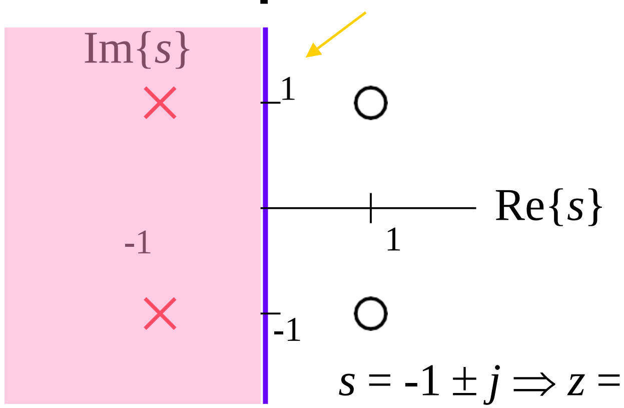
If some of poles are complex-valued, they may be paired together and **combined to form two-pole filter sections with real-valued coefficients.**

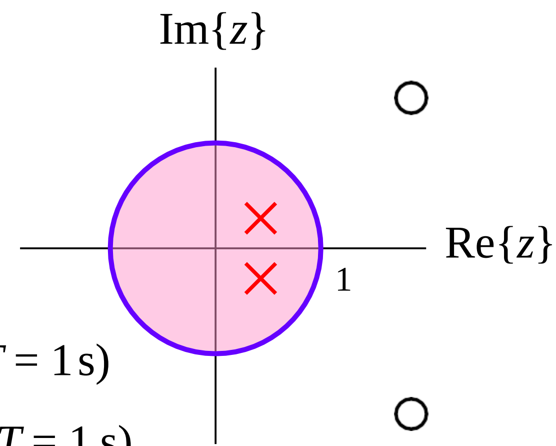
In addition, two factors containing real-valued poles may be **combined to form two-pole filters with real-valued coefficients.**

**Consequently, the resulting filter may be realized as a parallel bank of two-pole filters with real-valued coefficients.**

# Impulse Invariance Mapping

- Impulse invariance mapping is **z = e^sT^**



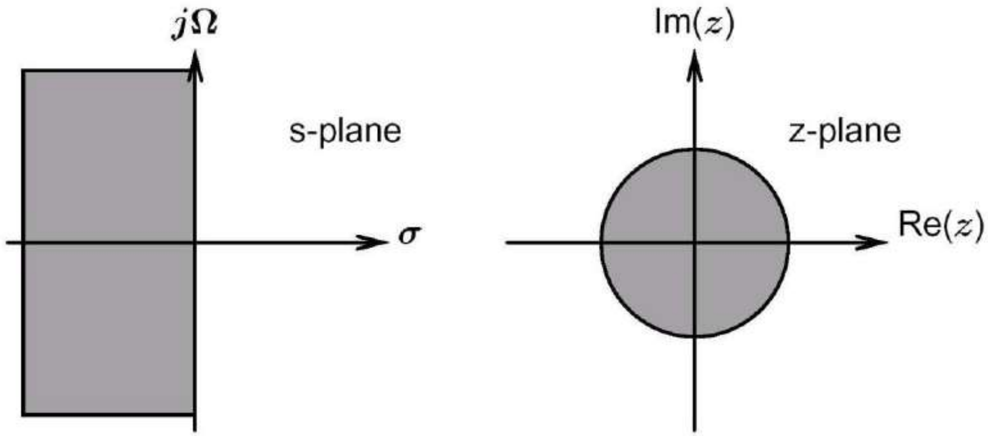

$$s = -1 \pm j \Rightarrow z = 0.198 \pm j\,0.31 \;(T = 1\text{ s})$$

$$s = 1 \pm j \;\Rightarrow z = 1.469 \pm j\,2.287 \;(T = 1\text{ s})$$

$s = j\,2\,\pi\,f$

| *Laplace Domain* | *Z Domain* |
|---|---|
| **Left-hand plane** | **Inside unit circle** |
| **Imaginary axis** | **Unit circle** |
| **Right-hand plane** | **Outside unit circle** |

# Discrete-Time IIR filter design




$H_c(s) \leftrightarrow H(z)$

1.Poles on the jΩ axis in the s-plane correspond to poles on the unit circle in the z-plane.

2.Poles in the left half of the s-plane correspond to poles inside the unit circle in the z-plane.

Hence stable and causal continuous-time filters will produce stable and causal discrete-time filters.

# Consequences:

a) $\sigma < 0 \rightarrow 0 < r < 1 \qquad \sigma > 0 \rightarrow r > 1 \qquad \sigma = 0 \rightarrow r = 1$

Then, the left-half of *p*-plane is mapped inside the unite circle in *z*-plane and right-half of *p*-plane is mapped into points that fall outside the unit circle in *z*-plane. This is one of the desirable properties of a good *p* -> *z* mapping.

b) $j\Omega$ -axis is mapped into the unit circle in *z*-plane as indicates above.

# Aliasing Effect:

When a continuous time signal $h_A(t)$ with spectrum $H_A(\Omega)$ is sampled with sampling frequency $\Omega_S = 2\pi F_S$ , the spectrum of the sampled signal is given by the following expressions:

$$H(j\Omega) = FT\left[h_A(nT)\right] = FT\left[h_A(n/F_S)\right]$$

$$H(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} H_A\left(j\Omega - jk\Omega_S\right)$$

$$H(f) = \frac{1}{T}\sum_{k=-\infty}^{\infty} H_A\left(f - kF_S\right) \quad where \quad f = \Omega/2\pi$$

Aliasing occurs if the sampling frequency $F_S$ is less then twice the highest frequency contained in $h_A$(t ).

## Aliasing effect impact:

1. The digital filter will possess (approximately) the frequency response characteristics of the corresponding analogue filter if the sampling interval $T$ is selected sufficiently small to avoid completely or minimize the effects of aliasing.

2. The impulse invariance method is inappropriate for designing high-pass filters or stop-band filters due to spectrum aliasing that results from the sampling process.

## Comments on $j\Omega$ -axis mapping: $\omega = \Omega T$

a) The mapping of $j\Omega$ -axis into the unit circle is not one-to-one.

b) $-\pi/T \le \Omega \le \pi/T \rightarrow -\pi \le \omega \le \pi$

c) Mapping of the adjacent strips - frequency interval:

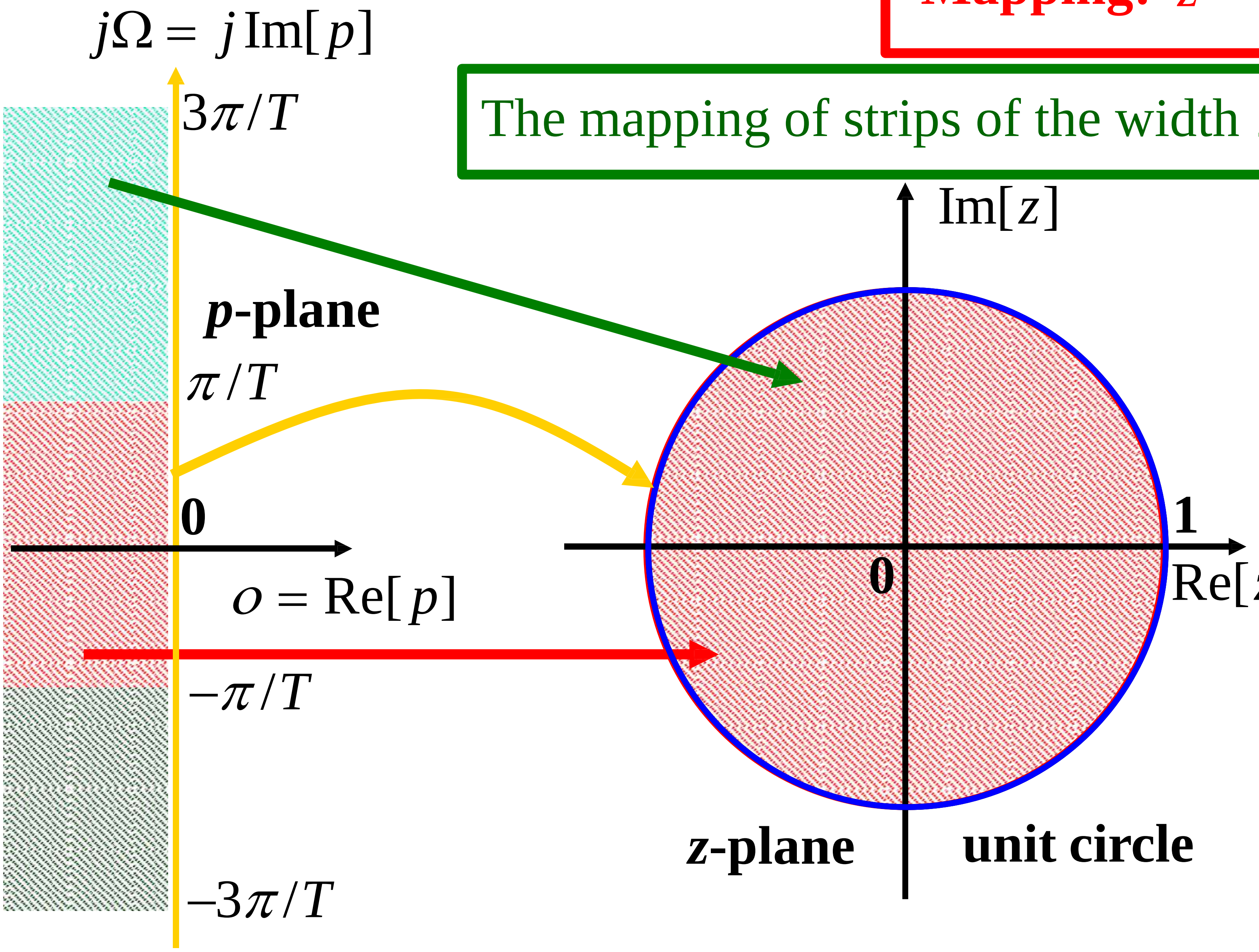
$$\pi/T \le \Omega \le 3\pi/T \rightarrow -\pi \le \omega \le \pi$$

d) General case:

$$(2k-1)\pi/T \le \Omega \le (2k+1)\pi/T \rightarrow -\pi \le \omega \le \pi$$

**Mapping:** $z = e^{pT}$

The mapping of strips of the width $2\pi$



**Summary on filter design with Impulse Invariant Method：**

1. Digital filter specification: $\omega_P, \omega_S, \delta_1 \; and \; \delta_2$

2. Transformation of requirements to the digital filter to the analogue filter:

$$\Omega = \omega / T \longrightarrow \quad \omega_P \to \Omega_p \quad \omega_S \to \Omega_S \quad \delta_1 \; and \; \delta_2$$

3. The analogue filter design: $H_A(p) = \sum_{k=1}^{N} \frac{c_k}{p + d_k}$

4. The analogue filter conversion to the digital filter:

$$H(z) = T \sum_{k=1}^{N} \frac{c_k}{1 - e^{-d_k T} z^{-1}}$$

## Comment on scaling factor application (*T*):

The frequency response of the filter obtained by impulse invariant transformation is given by

$$H(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} H_A\left(\Omega - k\Omega_S\right)$$

Under condition that

$$\left|H_A\left(\Omega - k\Omega_S\right)\right| \sim 0 \; for \; |k| > 0$$

we can obtain:

$$H(e^{j\omega}) \sim \frac{1}{T} H_A(\Omega)$$

If it is desired to get a digital filter with the same gain as the analogue filter possesses, it is necessary to transform the expression for *H(z)* originally given by

$$H(z) = \sum_{k=1}^{N} \frac{c_k}{1 - e^{-d_k T} z^{-1}}$$

in the form

$$H(z) = T \sum_{k=1}^{N} \frac{c_k}{1 - e^{-d_k T} z^{-1}}$$

Then:

$$H(e^{j\omega}) \sim H_A(\Omega)$$

# Butterworth Filter Design with Impulse Invariance

$$0.89125 \le |H(e^{j\omega})| \le 1, \quad 0 \le |\omega| \le 0.2\pi$$

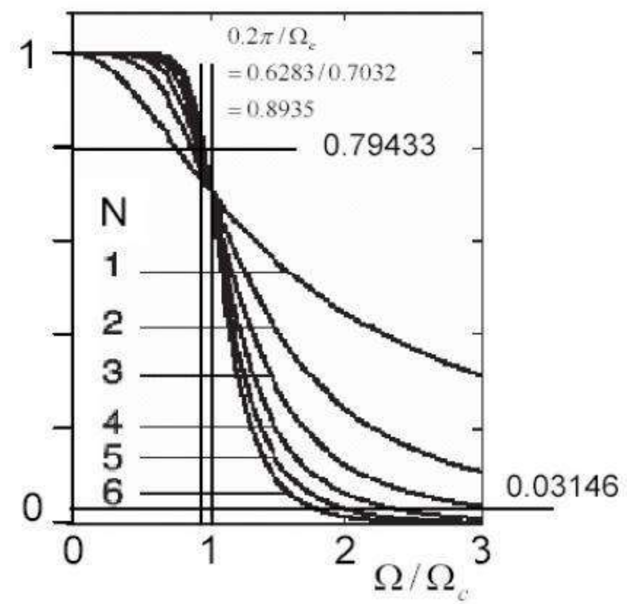
$$|H(e^{j\omega})| \le 0.17738, \quad 0.3\pi \le |\omega| \le \pi$$

What is $\Omega_c$ ? Value of $N$?

$$|H_c(j\Omega)|^2 = \frac{1}{1+(\Omega/\Omega_c)^{2N}} \quad \text{is monotonic}$$

$$|H_c(j\Omega)|^2 = \frac{1}{1+(0.2\pi/\Omega_c)^{2N}} = (0.89125)^2 = 0.79433$$

$$|H_c(j\Omega)|^2 = \frac{1}{1+(0.3\pi/\Omega_c)^{2N}} = (0.17738)^2 = 0.03146$$




$$(0.2\pi/\Omega_c)^{2N} = 1.25893 - 1 = 0.25893$$

$$\Rightarrow \quad 2N\{\log(0.2\pi) - \log\Omega_c\} = -0.58683 \quad \Rightarrow \quad 2N\{-0.20182 - \log\Omega_c\} = -0.58683$$

$$(0.3\pi/\Omega_c)^{2N} = 31.78269 - 1 = 30.78269$$

$$\Rightarrow \quad 2N\{\log(0.3\pi) - \log\Omega_c\} = 1.50219 \quad \Rightarrow \quad 2N\{-0.02573 - \log\Omega_c\} = 1.50219$$

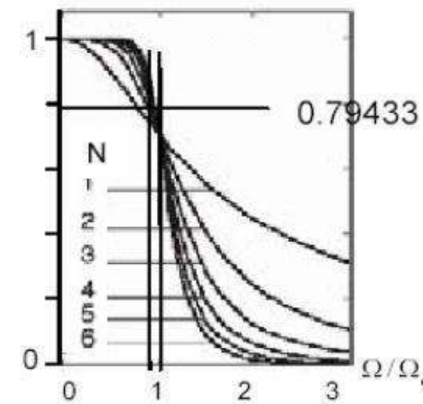
$$N \approx 5.9 \quad \Rightarrow \quad N = 6 \qquad -0.20182 - \log\Omega_c = -0.58683/12 \quad \Rightarrow \quad \Omega_c = 0.7032$$

# Butterworth Filter Design with Impulse Invariance

$$0.89125 \le |H(e^{j\omega})| \le 1, \quad 0 \le |\omega| \le 0.2\pi$$

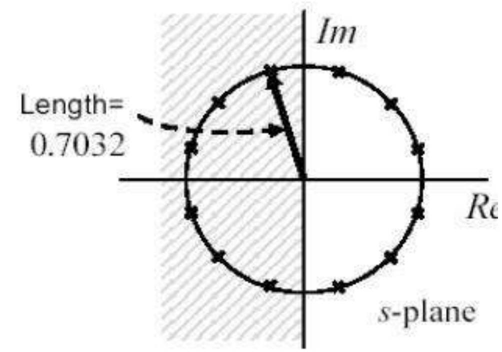
$$|H(e^{j\omega})| \le 0.17738, \quad 0.3\pi \le |\omega| \le \pi$$

$$H_c(s)H_c(-s) = \frac{1}{1+(s/j\Omega_c)^{12}} \qquad N = 6 \qquad \Omega_c = 0.7032$$




$$s_k = 0.7032 \bullet \exp\left\{\frac{j\pi}{12}(2k+5)\right\} \quad k = 0, 1, 2, \cdots, 11$$

$$H_c(s) = \frac{0.12093}{(s^2+0.3640s+0.4945)(s^2+0.9945s+0.4945)(s^2+1.3585s+0.4945)} = \sum_{k=1}^{N=6} \frac{A_k}{s-s_k}$$




$$h_c(t) = \begin{cases} \sum_{k=1}^{N=6} A_k e^{s_k t}, & t \ge 0 \\ 0, & t < 0 \end{cases}$$

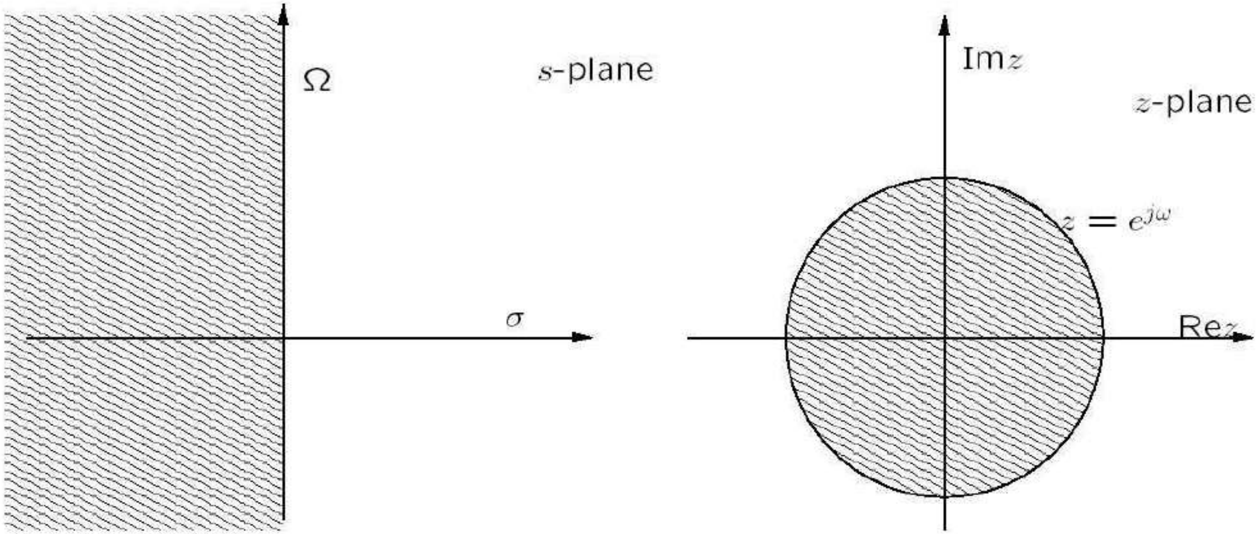
$$h[n] = T_d h_c(nT_d)$$

$$H(z) = \sum_{k=1}^{N} \frac{T_d A_k}{1 - e^{s_k T_d} z^{-1}}$$

# Bilinear Transform

To avoid aliasing, we need a one-to-one mapping from the s-plane to the z-plane.

- Map whole $\Omega$ axis to $-\pi \leq \omega \leq \pi$, or whole right $s$-plane to inside of unit circle in $z$-plane
- $s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$ or $z = \frac{1+sT/2}{1-sT/2}$

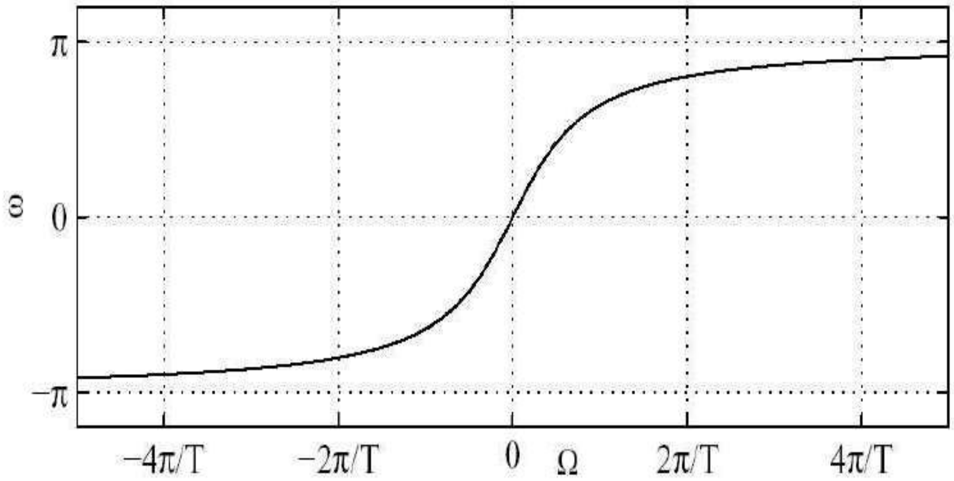


# Bilinear Transform

$$j\Omega = \frac{2}{T}\left(\frac{1-e^{j\omega}}{1+e^{j\omega}}\right) = j\frac{2}{T}\left(\frac{\sin\omega/2}{\cos\omega/2}\right) \Rightarrow \Omega = \frac{2}{T}\tan(\omega/2)$$

# Bilinear Transform

- $\Omega = \frac{2}{T} \tan \frac{\omega}{2}$ or $\omega = 2 \arctan \frac{\Omega T}{2}$
- Preserves stability, but not shape (warping of the frequency axis)



# Bilinear Transform

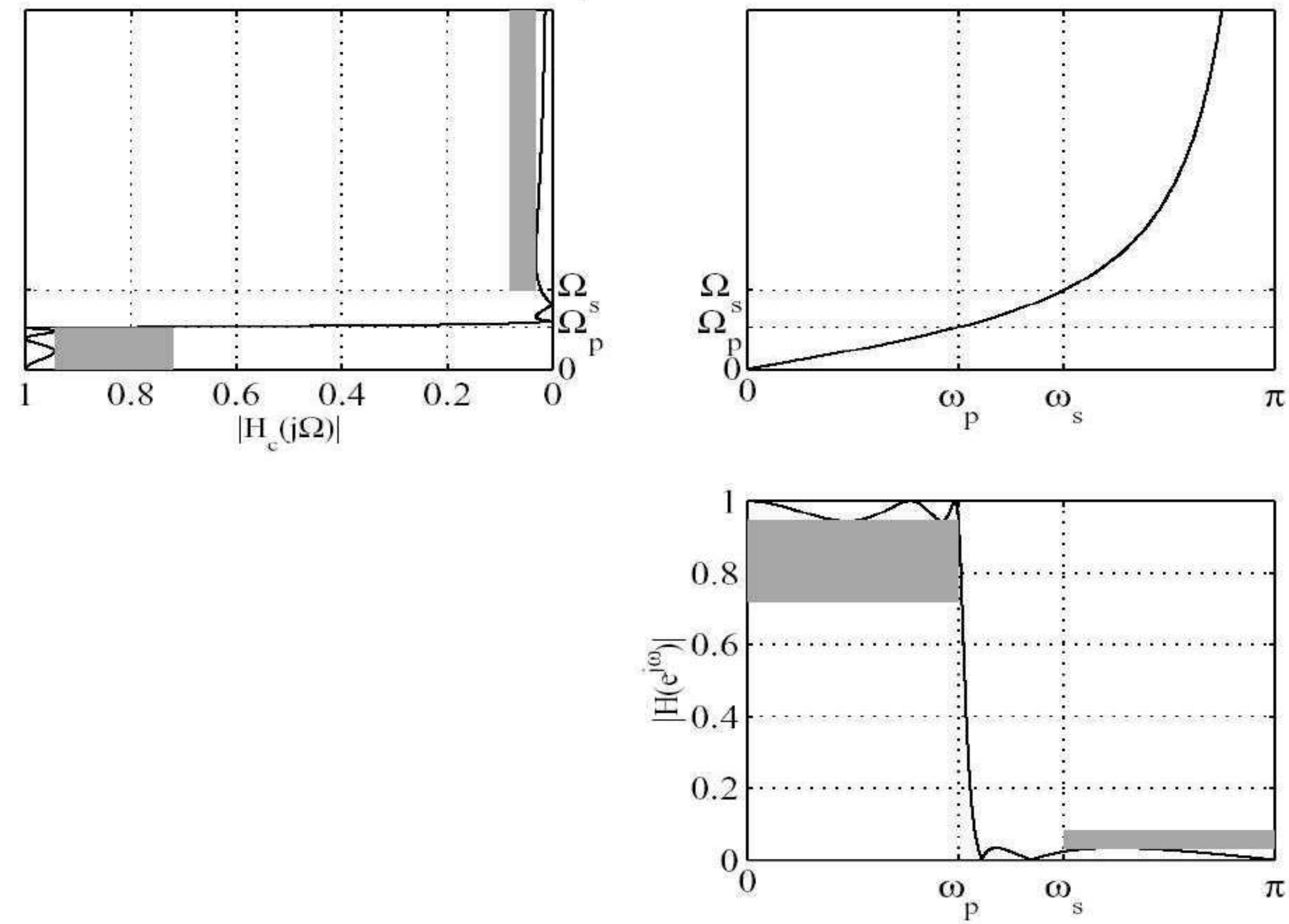
1. Perform frequency prewarp to obtain the corresponding analog filter specs (pick any $T$)

2. Design the analog filter $H_c(s)$ using any one of the analog filter prototypes.

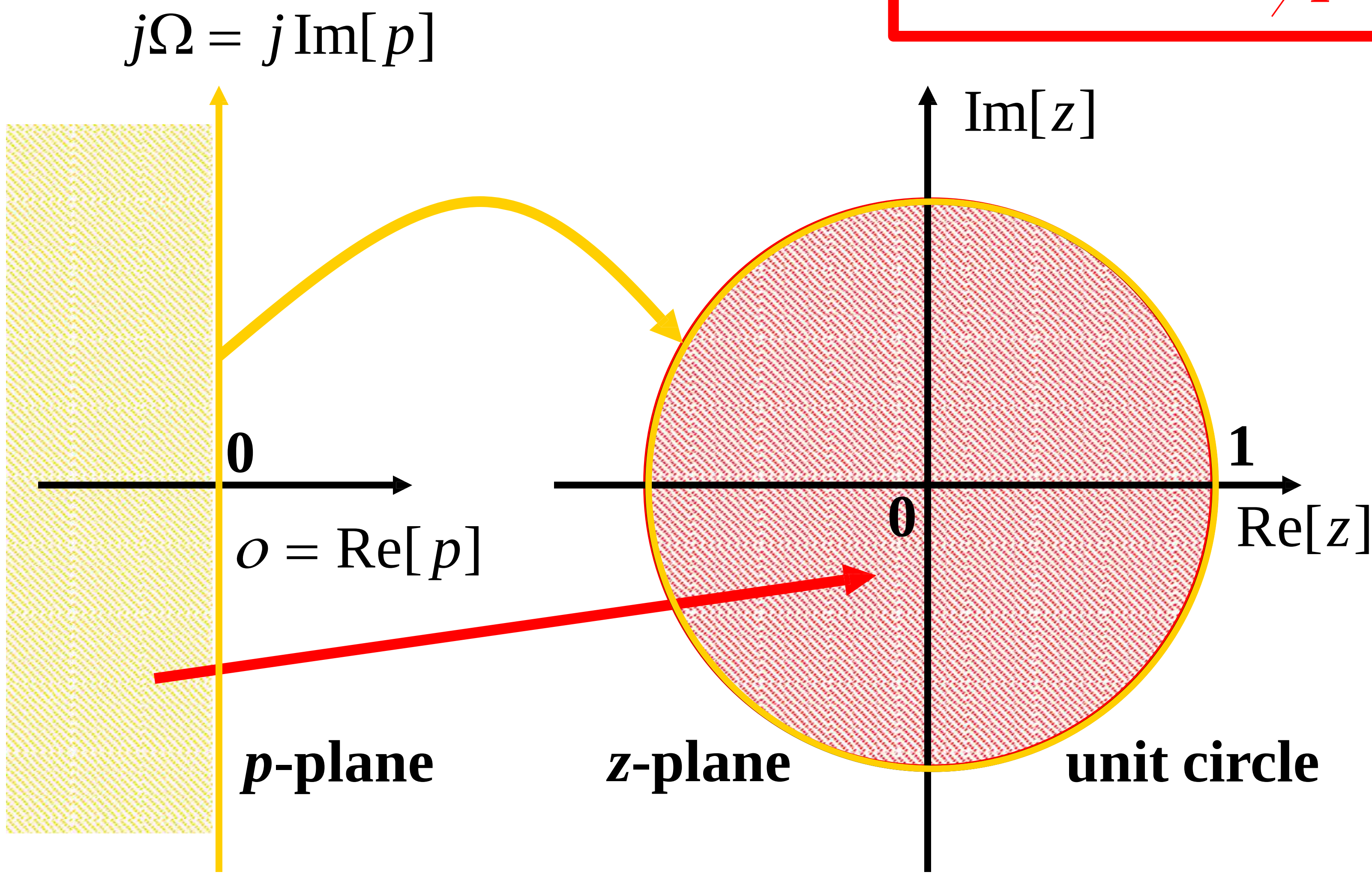
3. Transform $Hc(s)$ to $H(z)$.

# Bilinear Transform

Transformation of filters and specs:



**Mapping:** $z = \dfrac{2/T + p}{2/T - p}$



## Summary on digital filter design with Bilinear Transformation:

1. Digital filter specification: $\omega_P, \omega_S, \delta_1 \; and \; \delta_2$

2. Transformation of requirements to the digital filter to the analogue filter:

$$\Omega = \frac{2}{T}\tan\frac{\omega}{2} \longrightarrow \omega_P \to \Omega_p \quad \omega_S \to \Omega_S \qquad \delta_1 \; and \; \delta_2$$

3. The analogue filter design: $H_A(p)$

4. The analogue filter conversion to the digital filter:

$$H(z) = H_A(p)\Big|_{p=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}$$

# Comparision between Bilinear Transformation and Impulse Invariance